South East Asian J. of Mathematics and Mathematical Sciences
Vol. 21, No. 2 (2025), pp. 81-94

DOI: 10.56827/SEAJMMS.2025.2102.6 **ISSN (Online): 2582-0850**

ISSN (Print): 0972-7752

A STUDY ON $(\mathcal{L}cs)_n$-MANIFOLDS INDUCED WITH $\mathcal{SVK}$-CONNECTION

N. V. C. Shukla and Amisha Sharma

Department of Mathematics and Astronomy,
University of Lucknow, Lucknow, INDIA

E-mail : nvcshukla72@gmail.com, amishasharma966@gmail.com

(**Received:** *Feb. 27, 2025* **Accepted:** *Jul. 15, 2025* **Published:** *Aug. 30, 2025)*

Abstract: The present paper aims to define and discuss $(\mathcal{L}cs)_n$-manifolds and Schouten van Kampen connection. We discussed the curvature tensor and Ricci curvature tensor of this manifold with respect to the $\mathcal{SVK}$-connection. We studied conformally flat, projectively flat, conharmonically flat, and concircularly flat $(\mathcal{L}cs)_n$-manifolds with the $\mathcal{SVK}$-connection. At last, we gave an example of $(\mathcal{L}cs)_n$-manifolds with $\mathcal{SVK}$-connection.

Keywords and Phrases: $(\mathcal{L}cs)_n$-manifold, $\mathcal{SVK}$-connection.

2020 Mathematics Subject Classification: 53C15, 53C20.

1. Introduction

In 1989, K. Matsumoto [6] introduced the notion of Lorentzian para-Sasakian manifolds and the generalization of LP-Sasakian manifolds. Lorentzian concircular structure manifolds (shortly, $(\mathcal{L}cs)_n$-manifolds) were introduced in 2003 by A. A. Shaikh [9]. In 2005 and 2006, Shaikh and Baishya [10], [11] investigated the application of $(\mathcal{L}cs)_n$-manifolds to the general theory of relativity and cosmology. $(\mathcal{L}cs)_n$-manifolds are also studied by Atceken et al ([1], [2]), D Narain and S, Yadav [13].

The $\mathcal{SVK}$-connection, endowed with an affine connection, is one of the most natural connections adapted to a pair of distributions on a differentiable manifold [3], [8]. Solov'ev [12] investigated hyperdistributions in Riemannian manifolds using the $\mathcal{SVK}$-connection. Then, Zbigniew Olszak [7] studied the Schouten-van Kampen affine connection in relation to an almost contact metric structure and obtained certain curvature properties of this connection on the manifold. Ahmet Yildiz [14], Semra Zeren [15], M. Manev [5], and others also studied $\mathcal{SVK}$-connection on different manifolds such as f-Kenmotsu manifolds, LP-Sasakian manifolds.

2. Preliminaries

In a Lorentzian manifold $(\mathcal{M}, g)$, we define a vector field ρ and 1-form $\mathcal{A}$ as

$$\mathcal{A}(X) = g(X, \rho), \tag{2.1}$$

for any vector field $X \in \chi(\mathcal{M})$. This ρ is called a concircular vector field [15] if

$$(\bigtriangledown_{X_1}\mathcal{A})(X_2) = \alpha\{g(X_1, X_2) + \omega(X_1)A(X_2)\}, \tag{2.2}$$

where α is a non-zero scalar function and ω is a closed 1-form.
Let $\mathcal{M}$ be a Lorentzian manifold admitting a unit timelike concircular vector field ξ, called the generator of the manifold, Then we have

$$g(\xi, \xi) = -1 \tag{2.3}$$

Since ξ is the unit concircular vector field, there exists a non-zero 1-form η so that for

$$g(X, \xi) = \eta(X) \tag{2.4}$$

following equations

$$(\bigtriangledown_{X_1}\eta)(X_2) = \alpha[g(X_1, X_2) + \eta(X_1)\eta(X_2)] \tag{2.5}$$

and

$$\bigtriangledown_X\xi = \alpha[X + \eta(X)\xi] \tag{2.6}$$

hold for all vector fields X, X_1, X_2 on $\mathcal{M}$, where $\bigtriangledown$ denotes the covariant derivative with respect to Lorentzian metric g and α is non-zero scalar function satisfying

$$\bigtriangledown_X\alpha = (X\alpha) = d\alpha(X) = \rho\eta(X), \tag{2.7}$$

where ρ is a certain scalar function given by ρ=-($\xi\alpha$).
If we put

$$\phi X = \frac{1}{\alpha}\bigtriangledown_X \xi, \tag{2.8}$$

Then from (2.5) and (2.8) we say that

$$\phi X = X + \eta(X)\xi, \tag{2.9}$$

from which we have that ϕ is a symmetric $(1,1)$ tensor field and is called the structure tensor of the manifold. Thus the Lorentzian manifold $\mathcal{M}$ together with unit timelike concircular vector field ξ, it's associate 1-form η and (1,1)-tensor field ϕ, is said to be Lorentzian concircular structure manifold $(\mathcal{M}, g, \phi, \xi, \eta, \alpha)$(briefly, $(\mathcal{L}cs)_n$-manifold) [9]. In particular if α=1, we obtain LP-Sasakian structure manifold [4].

A differentiable manifold $\mathcal{M}$ of dimension n is known to be $(\mathcal{L}cs)_n$-manifold if it admits a $(1,1)$-type tensor field ϕ, a covariant vector field η and a Lorentzian metric g which satisfy

$$\begin{aligned} \phi^2 X =& X + \eta(X)\xi, \\ \eta \circ \phi =& 0, \\ \phi\xi =& 0, \\ \eta(\xi) =& -1, \\ g(\phi X_1, \phi X_2) =& g(X_1, X_2) + \eta(X_1)\eta(X_2) \end{aligned} \tag{2.10}$$

for all $X_1, X_2 \in \chi(\mathcal{M})$. In an $(\mathcal{L}cs)_n$-manifold $\mathcal{M}$, we have the following relations

$$(\bigtriangledown_{X_1}\phi)X_2 = \alpha[g(X_1, X_2)\xi + 2\eta(X_1)\eta(X_2)\xi + \eta(X_2)X_1] \tag{2.11}$$

$$\eta(\mathcal{R}(X_1, X_2)X_3) = (\alpha^2 - \rho)[g(X_2, X_3)\eta(X_1) - g(X_1, X_3)\eta(X_2)], \tag{2.12}$$

$$\mathcal{R}(X_1, X_2)\xi = (\alpha^2 - \rho)[\eta(X_2)X_1 - \eta(X_1)X_2], \tag{2.13}$$

$$\mathcal{R}(\xi, X)\xi = (\alpha^2 - \rho)(\eta(X)\xi + X), \tag{2.14}$$

$$\mathcal{S}(X, \xi) = (n-1)(\alpha^2 - \rho)\eta(X). \tag{2.15}$$

$$QX = (n-1)(\alpha^2 - \rho)X. \tag{2.16}$$

$$\mathcal{S}(\phi X_1, \phi X_2) = (n-1)(\alpha^2 - \rho)g(\phi X_1, \phi X_2). \tag{2.17}$$

for any vector fields $X, X_1, X_2, X_3 \in \chi(\mathcal{M})$. $\mathcal{R}$ being the curvature tensor and $\mathcal{S}$ being the Ricci tensor.

Definition 2.1. *An n-dimensional pseudo-Riemannian manifold is said to be an η-Einstein manifold if the Ricci tensor of the manifold satisfies the condition*

$$\mathcal{S}(X_1, X_2) = ag(X_1, X_2) + b\eta(X_1)\eta(X_2). \tag{2.18}$$

If $b = 0$, the manifold is an Einstein manifold.
If $a = 0$, the manifold is a special type of η-Einstein manifold.

3. Schouten-van Kampen connection

Let U_1 and U_2 be two complementary distributions on a connected pseudo-Riemannian manifold of an arbitrary signature $(p, n-p), 0 \leq p \leq n, n = dim\mathcal{M} \geq 2$, such that $dim(U_1) = (n-1)$ and $dim(U_2) = 1$ and the distribution U_2 is non-null. Then we have $TM = U_1 \otimes U_2$, Also $U_1 \cap U_2 = \{0\}$ and $U_1 \perp U_2$ such that

$$U_1 = ker\eta, \quad U_2 = span\{\xi\}$$

where η is a linear form and ξ is unit vector field such that $\eta(X) = \epsilon g(X, \xi)$. Moreover, it holds that $\bigtriangledown_X \xi \in U_1$.
For any $X \in TM$, if X^{u_1} and X^{u_2} are the projections of X onto U_1 and U_2.
We have

$$X = X^{u_1} + X^{u_2}$$

with

$$X^{u_1} = X - \eta(X)\xi,$$

and

$$X^{u_2} = \eta(X)\xi.$$

The $\mathcal{SVK}$-connection $\bar{\bigtriangledown}$ associated with Levi-Civita connection $\bigtriangledown$ and adapted to the pair of the distribution (U_1, U_2) is defined by [3]

$$\bar{\bigtriangledown}_{X_1} X_2 = (\bigtriangledown_{X_1} X_2^{u_1})^{u_1} + (\bigtriangledown_{X_1} X_2^{u_2})^{u_2}. \tag{3.1}$$

From the above equation, we have

$$(\bigtriangledown_{X_1} X_2^{u_1})^{u_1} = \bigtriangledown_{X_1} X_2 - \eta(\bigtriangledown_{X_1} X_2)\xi - \eta(X_2) \bigtriangledown_{X_1} \xi. \tag{3.2}$$

and

$$(\bigtriangledown_{X_1} X_2^{u_2})^{u_2} = (\bigtriangledown_{X_1} \eta)(X_2)\xi + \eta(\bigtriangledown_{X_1} X_2)\xi. \tag{3.3}$$

Thus, the $\mathcal{SVK}$-connection with the help of Levi-Civita connection is being expressed as [8]

$$\bar{\bigtriangledown}_{X_1} X_2 = \bigtriangledown_X X_2 - \eta(X_2) \bigtriangledown_{X_1} \xi + (\bigtriangledown_{X_1} \eta)(X_2)\xi. \tag{3.4}$$

Thus, with the help of $\mathcal{SVK}$-connection, one can characterize many properties of some geometric objects, which are connected with the distributions U_1, U_2. We compute that g, ξ, η are parallel with respect to $\bar{\bigtriangledown}$, that is $\bar{\bigtriangledown} g = 0$, $\bar{\bigtriangledown}\xi = 0$, $\bar{\bigtriangledown}\eta = 0$. Also the torsion $\bar{T}$ of connection $\bar{\bigtriangledown}$ is given as [12]

$$\bar{T}(X_1, X_2) = \eta(X_1) \bigtriangledown_{X_2} \xi - \eta(X_2) \bigtriangledown_{X_1} \xi - 2d\eta(X_1, X_2)\xi. \tag{3.5}$$

4. Curvature tensor on $(\mathcal{L}cs)_n$-manifolds with $\mathcal{SVK}$-connection

Let $\mathcal{M}$ be an $(\mathcal{L}cs)_n$-manifold with $\mathcal{SVK}$-connection. Then using (2.5), (2.6) in (3.2), we have the $\mathcal{SVK}$-connection $\bar{\nabla}$ associated to the Levi-Civita connection ∇ as

$$\bar{\nabla}_{X_1}X_2 = \nabla_{X_1}X_2 - \alpha\eta(X_2)X_1 + \alpha g(X_1, X_2)\xi \tag{4.1}$$

Let $\mathcal{R}$ and $\bar{\mathcal{R}}$ be curvature tensors of Levi-Civita connection ∇ and the $\mathcal{SVK}$-connection $\bar{\nabla}$,

$$\bar{\mathcal{R}}(X_1, X_2)X_3 = \bar{\nabla}_{X_1}\bar{\nabla}_{X_2}X_3 - \bar{\nabla}_{X_2}\bar{\nabla}_{X_1}X_3 - \bar{\nabla}_{[X_1,X_2]}X_3 \tag{4.2}$$

Using (4.1), together with (2.5), (2.6), and (2.10), we obtain

$$\begin{aligned}\bar{\nabla}_{X_1}\bar{\nabla}_{X_2}X_3 =& \nabla_{X_1}\nabla_{X_2}X_3 + (\rho + 2\alpha^2)\eta(X_1)g(X_2, X_3)\xi + \alpha(X_1 g(X_2, X_3))\xi \\ &+ 2\alpha^2 g(X_2, X_3)X_1 - \alpha(X_1\eta(X_3))X_2 - \alpha\eta(X_3)\nabla_{X_1}X_2 \\ &+ \alpha g(X_1, \nabla_{X_2}X_3)\xi - \alpha\eta(\nabla_{X_2}X_3)X_1 + \alpha^2\eta(X_3)\eta(X_2)X_1 \\ &- \rho\eta(X_3)\eta(X_1)X_2,\end{aligned} \tag{4.3}$$

$$\begin{aligned}\bar{\nabla}_{X_2}\bar{\nabla}_{X_1}X_3 =& \nabla_{X_2}\nabla_{X_1}X_3 + (\rho + 2\alpha^2)\eta(X_2)g(X_1, X_3)\xi + \alpha(X_2 g(X_1, X_3))\xi \\ &+ 2\alpha^2 g(X_1, X_3)X_2 - \alpha(X_2\eta(X_3))X_1 - \alpha\eta(X_3)\nabla_{X_2}X_1 \\ &+ \alpha g(X_2, \nabla_{X_1}X_3)\xi - \alpha\eta(\nabla_{X_1}X_3)X_2 + \alpha^2\eta(X_3)\eta(X_1)X_2 \\ &- \rho\eta(X_3)\eta(X_2)X_1,\end{aligned} \tag{4.4}$$

$$\begin{aligned}\bar{\nabla}_{[X_1,X_2]}X_3 =& \nabla_{[X_1,X_2]}X_3 + \alpha g(\nabla_{X_1}X_2, X_3)\xi - \alpha g(\nabla_{X_2}X_1, X_3)\xi \\ &- \alpha\eta(X_3)\nabla_{X_1}X_2 + \alpha\eta(X_3)\nabla_{X_2}X_1.\end{aligned} \tag{4.5}$$

Using (4.3), (4.4) and (4.5) in (4.2), we have

$$\begin{aligned}\bar{\mathcal{R}}(X_1, X_2)X_3 =& \mathcal{R}(X_1, X_2)X_3 + 3\alpha^2[g(X_2, X_3)X_1 - g(X_1, X_3)X_2] \\ &+ (\rho + 2\alpha^2)[\eta(X_1)g(X_2, X_3)\xi - \eta(X_2)g(X_1, X_3)\xi \\ &+ \eta(X_2)\eta(X_3)X_1 - \eta(X_1)\eta(X_3)X_2].\end{aligned} \tag{4.6}$$

By putting $X_3 = \xi$ in (4.6), we have

$$\bar{\mathcal{R}}(X_1, X_2)\xi = 2(\alpha^2 - \rho)(\eta(X_2)X_1 - \eta(X_1)X_2). \tag{4.7}$$

Taking inner product with ξ in (4.6), we have

$$\eta(\bar{\mathcal{R}}(X_1, X_2)X_3) = 2(\alpha^2 - \rho)(g(X_2, X_3)\eta(X_1) - g(X_1, X_3)\eta(X_2)). \tag{4.8}$$

On contracting (4.6), we have Ricci tensor $\bar{\mathcal{S}}$ of an $(\mathcal{L}cs)_n$-manifold with respect to the $\mathcal{SVK}$-connection $\bar{\nabla}$ as

$$\bar{\mathcal{S}}(X_1, X_2) = \mathcal{S}(X_1, X_2) + [\alpha^2(3n-5) - \rho)]g(X_1, X_2) + (n-2)(2\alpha^2 + \rho)\eta(X_1)\eta(X_2). \tag{4.9}$$

$$\bar{\mathcal{S}}(X_1, \xi) = 2(n-1)(\alpha^2 - \rho)\eta(X_1). \tag{4.10}$$

Again the Ricci operator $\bar{\mathcal{Q}}$ of an $(\mathcal{L}cs)_n$-manifold with respect to the $\mathcal{SVK}$-connection $\bar{\nabla}$ is given as

$$\bar{\mathcal{Q}}X_1 = \mathcal{Q}X_1 + [\alpha^2(3n-5) - \rho)]X_1 + (n-2)(2\alpha^2 + \rho)\eta(X_1)\xi. \tag{4.11}$$

where $\bar{\mathcal{S}}(X_1, X_2) = g(\bar{\mathcal{Q}}X_1, X_2)$.

Now, contracting (4.9) with respect to X_1 and X_2, we have the scalar curvature $\bar{r}$ of the $(\mathcal{L}cs)_n$-manifold with respect to the $\mathcal{SVK}$-connection $\bar{\nabla}$ as

$$\bar{r} = r + (n-1)(3n-4)\alpha^2 - 2(n-1)\rho. \tag{4.12}$$

where r is scalar curvature with respect to Levi-Civita connection ∇.

5. Projectively flat $(\mathcal{L}cs)_n$-manifolds with $\mathcal{SVK}$-connection

The projective curvature tensor $\bar{P}$ with respect to the $\mathcal{SVK}$-connection $\bar{\nabla}$ is defined as

$$\bar{P}(X_1, X_2)X_3 = \bar{\mathcal{R}}(X_1, X_2)X_3 - \frac{1}{n-1}[\bar{\mathcal{S}}(X_2, X_3)X_1 - \bar{\mathcal{S}}(X_1, X_3)X_2]. \tag{5.1}$$

A manifold is said to be projectively flat if

$$\bar{P}(X_1, X_2)X_3 = 0. \tag{5.2}$$

From equations (5.1) and (5.2), we have

$$\bar{\mathcal{R}}(X_1, X_2)X_3 = \frac{1}{n-1}[\bar{\mathcal{S}}(X_2, X_3)X_1 - \bar{\mathcal{S}}(X_1, X_3)X_2]. \tag{5.3}$$

Taking inner product in (5.3) with ξ, we get

$$g(\bar{\mathcal{R}}(X_1, X_2)X_3, \xi) = \frac{1}{n-1}[\bar{\mathcal{S}}(X_2, X_3)g(X_1, \xi) - \bar{\mathcal{S}}(X_1, X_3)g(X_2, \xi)]. \tag{5.4}$$

Using (4.8) and (5.4), we get

$$2(\alpha^2 - \rho)(g(X_2, X_3)\eta(X_1) - g(X_1, X_3)\eta(X_2)) = \frac{1}{n-1}[\bar{\mathcal{S}}(X_2, X_3)\eta(X_1) - \bar{\mathcal{S}}(X_1, X_3)\eta(X_2)].$$

Putting $X_1 = \xi$, the above equation takes form

$$2(\alpha^2 - \rho)[-g(X_2, X_3)) - \eta(X_2)\eta(X_3)] = \frac{1}{n-1}[-\bar{\mathcal{S}}(X_2, X_3) -2(n-1)(\alpha^2 - \rho)\eta(X_2)\eta(X_3)].$$

Solving, we have the following:

Theorem 5.1. *A projectively flat $(\mathcal{L}cs)_n$-manifold with $\mathcal{SVK}$-connection is an Einstein manifold.*

An $(\mathcal{L}cs)_n$-manifolds with $\mathcal{SVK}$-connection is said to be ϕ-projectively flat if

$$'\bar{P}(\phi X_1, \phi X_2, \phi X_3, \phi X_4) = 0, \tag{5.5}$$

where $X_1, X_2, X_3, X_4 \in \chi(\mathcal{M})$.

By (5.1), we have

$$'\bar{P}(X_1, X_2, X_3, X_4) = {}'\bar{\mathcal{R}}(X_1, X_2, X_3, X_4) - \frac{1}{n-1}[\bar{\mathcal{S}}(X_2, X_3)g(X_1, X_4) - \bar{\mathcal{S}}(X_1, X_3)g(X_2, X_4)],$$

where $'\bar{P}(X_1, X_2, X_3, X_4) = g(\bar{P}(X_1, X_2)X_3, X_4)$ and $'\bar{\mathcal{R}}(X_1, X_2, X_3, X_4) = g(\bar{\mathcal{R}}(X_1, X_2)X_3, X_4)$.

Now putting $X_1 = \phi X_1, X_2 = \phi X_2, X_3 = \phi X_3, X_4 = \phi X_4$ in above and using (5.5), we have

$$'\bar{\mathcal{R}}(\phi X_1, \phi X_2, \phi X_3, \phi X_4) = \frac{1}{n-1}[\bar{\mathcal{S}}(\phi X_2, \phi X_3)g(\phi X_1, \phi X_4) - \bar{\mathcal{S}}(\phi X_1, \phi X_3)g(\phi X_2, \phi X_4)]. \tag{5.6}$$

Let $\{\phi e_1, \phi e_2.....\phi e_{n-1}, \xi\}$ be the local orthonormal basis of vector fields in M. Putting $X_1 = X_4 = e_i$ in (5.6) and taking summation over $i = 1$ to $n-1$, we have

$$\sum_{i=1}^{n-1} {}'\bar{\mathcal{R}}(\phi e_1, \phi X_2, \phi X_3, \phi e_4) = \frac{1}{n-1}[\bar{\mathcal{S}}(\phi X_2, \phi X_3)\sum_{i=1}^{n-1} g(\phi e_1, \phi e_4) - \sum_{i=1}^{n-1} \bar{\mathcal{S}}(\phi e_1, \phi X_3)g(\phi X_2, \phi e_4)]. \tag{5.7}$$

Solving the above, we have

$$\bar{\mathcal{S}}(\phi X_2, \phi X_3) = \frac{1}{n-1}[(n-1)\bar{\mathcal{S}}(\phi X_2, \phi X_3) - g(\bar{\mathcal{Q}}\phi X_2, \phi X_3)],$$

which yields

$$\bar{\mathcal{S}}(\phi X_2, \phi X_3) = 0 = g(\bar{Q}\phi X_2, \phi X_3). \tag{5.8}$$

Using $\bar{\mathcal{Q}}\phi = \phi\bar{\mathcal{Q}} = \phi\mathcal{Q} + [(3n-5)\alpha^2 - \rho]$ in (5.8)

$$g(\phi\mathcal{Q}X_2, \phi X_3) + [(3n-5)\alpha^2 - \rho]g(\phi X_2, \phi X_3) = 0. \tag{5.9}$$

Using (2.10) and (2.16), (5.9) reduces to

$$S(X_2, X_3) = [\rho - (3n-5)\alpha^2]g(X_2, X_3) + [n\rho - 2(2n-3)\alpha^2]\eta(X_2)\eta(X_3). \tag{5.10}$$

From (4.9) and (5.10), we obtain

$$\bar{\mathcal{S}}(X_2, X_3) = 2(n-1)(\rho - \alpha^2)\eta(X_2)\eta(X_3). \tag{5.11}$$

Thus, we state the following

Theorem 5.2. *A ϕ-projectively flat $(\mathcal{L}cs)_n$-manifold with $\mathcal{SVK}$-connection is a special type of η-Einstein manifold.*

6. Conformally flat $(\mathcal{L}cs)_n$-manifolds with $\mathcal{SVK}$-connection

The Weyl conformal curvature tensor $\bar{C}$ of type $(1,3)$ of an n-dimensional Riemannian manifold is given as

$$\begin{aligned}\bar{C}(X_1, X_2)X_3 =& \mathcal{R}(X_1, X_2)X_3 \\ &- \frac{1}{(n-2)}\big[\bar{\mathcal{S}}(X_2, X_3)X_1 - \bar{\mathcal{S}}(X_1, X_3)X_2 \\ &+ g(X_2, X_3)\bar{\mathcal{Q}}X_1 - g(X_1, X_3)\bar{\mathcal{Q}}X_2\big] \\ &+ \frac{r}{(n-1)(n-2)}\big[g(X_2, X_3)X_1 - g(X_1, X_3)X_2\big]\end{aligned} \tag{6.1}$$

Let us suppose that $(\mathcal{L}cs)_n$-manifold is Conformally flat with respect to the $\mathcal{SVK}$-connection, we have

$$\bar{C}(X_1, Y)X_3 = 0. \tag{6.2}$$

Using equation (6.1) and (6.2)

$$\begin{aligned}\bar{\mathcal{R}}(X_1, X_2)X_3 =& \frac{1}{(n-2)}\big[\bar{\mathcal{S}}(X_2, X_3)X_1 - \bar{\mathcal{S}}(X_1, X_3)Y \\ &+ g(X_2, X_3)\bar{\mathcal{Q}}X - g(X_1, X_3)\bar{\mathcal{Q}}X_2\big] \\ &- \frac{\bar{r}}{(n-1)(n-2)}\big[g(X_2, X_3)X_1 - g(X_1, X_3)X_2\big].\end{aligned} \tag{6.3}$$

Taking inner product with X_4

$$\begin{aligned} g(\bar{R}(X_1, X_2)X_3, X_4) =& \frac{1}{(n-2)}[\bar{\mathcal{S}}(X_2, X_3)g(X_1, X_4) - \bar{\mathcal{S}}(X_1, X_3)g(X_2, X_4) \\ &+ g(X_2, X_3)\bar{\mathcal{S}}(X_1, X_4) - g(X_1, X_3)\bar{\mathcal{S}}(X_2, X_4)] \\ &- \frac{\bar{r}}{(n-1)(n-2)}\big[g(X_2, X_3)g(X_1, X_4) \\ &- g(X_1, X_3)g(X_2, X_4)\big]. \end{aligned} \tag{6.4}$$

Putting $X_4 = \xi$ and using equations (2.4) and (4.10), we have

$$\begin{aligned} \eta(\bar{R}(X_1, X_2)X_3) =& \frac{1}{(n-2)}[\bar{\mathcal{S}}(X_2, X_3)\eta(X_1) - \bar{\mathcal{S}}(X_1, X_3)\eta(X_2)] \\ &+ [\frac{2(n-1)(\alpha^2 - \rho)}{n-2} - \frac{\bar{r}}{(n-1)(n-2)}] \\ &[g(X_2, X_3)\eta(X_1) - g(X_1, X_3)\eta(X_2)]. \end{aligned} \tag{6.5}$$

Using equation (4.8), above equation becomes

$$\begin{aligned} \bar{\mathcal{S}}(X_2, X_3)\eta(X_1) =& \bar{\mathcal{S}}(X_1, X_3)\eta(X_2) + [\frac{\bar{r}}{(n-1)} - 2(\alpha^2 - \rho)] \\ &[g(X_2, X_3)\eta(X_1) - g(X_1, X_3)\eta(X_2)]. \end{aligned} \tag{6.6}$$

Using $X = \xi$ in (6.6) and using equations (4.10) and (2.3), we have

$$\begin{aligned} \bar{\mathcal{S}}(X_2, X_3) =& [\frac{\bar{r}}{(n-1)} - 2(\alpha^2 - \rho)]g(X_2, X_3) \\ &+ [\frac{\bar{r}}{(n-1)} - 2n(\alpha^2 - \rho)]\eta(X_2)\eta(X_3). \end{aligned} \tag{6.7}$$

Hence, we can state the following:

Theorem 6.1. *A conformally flat $(\mathcal{L}cs)_n$-manifold with $\mathcal{SVK}$-connection is an η-Einstein manifold.*

7. Conharmonically flat $(\mathcal{L}cs)_n$-manifolds with $\mathcal{SVK}$-connection

The conharmonic curvature tensor $\bar{V}$ with respect to the $\mathcal{SVK}$-connection $\bar{\bigtriangledown}$ is defined as

$$\begin{aligned} \bar{V}(X_1, X_2)X_3 =& \bar{\mathcal{R}}(X_1, X_2)X_3 - \big[\bar{\mathcal{S}}(X_2, X_3)X_1 - \bar{\mathcal{S}}(X_1, X_3)X_2 \\ &+ g(X_2, X_3)\bar{Q}X_1 - g(X_1, X_3)\bar{Q}X_2\big]. \end{aligned} \tag{7.1}$$

A manifold is said to be conharmonically flat if

$$\bar{V}(X_1, X_2)X_3 = 0. \tag{7.2}$$

From equations (7.1) and (7.2), we have

$$\bar{\mathcal{R}}(X_1, X_2)X_3 = \bar{\mathcal{S}}(X_2, X_3)X - \bar{\mathcal{S}}(X_1, X_3)X_2 + g(X_2, X_3)\bar{Q}X_1 - g(X_1, X_3)\bar{Q}X_2. \tag{7.3}$$

Now using (4.6), (4.9) and (4.11)

$$\begin{aligned}\mathcal{R}(X_1, X_2)X_3 =& \mathcal{S}(X_2, X_3)X_1 - \mathcal{S}(X_1, X_3)X_2\\ &+ (\alpha^2(6n-13) - 2\rho)[g(X_2, X_3)X_1 - g(X_1, X_3)X_2]\\ &+ (n-3)(2\alpha^2 + \rho)[g(X_2, X_3)\eta(X_1)\xi - g(X_1, X_3)\eta(X_2)\xi]\\ &+ (n-3)(2\alpha^2 + \rho)[\eta(X_2)\eta(X_3)X_1 - \eta(X_1)\eta(X_3)X_2]\\ &+ [g(X_2, X_3)QX_1 - g(X_1, X_3)QX_2].\end{aligned} \tag{7.4}$$

Put $X_1 = \xi$ and using (2.13),(2.15) and (2.16) in above, we get

$$\begin{aligned}(\alpha^2 - \rho)[g(X_2, X_3)\xi - \eta(X_3)X_2] =& \mathcal{S}(X_2, X_3)\xi + \big(5n\alpha^2 - 8\alpha^2 - 2\rho(1-n)\big)\\ &\times g(X_2, X_3)\xi.\end{aligned} \tag{7.5}$$

Taking inner product with ξ, we have

$$\begin{aligned}\mathcal{S}(X_2, X_3) =& (9\alpha^2 - 5n\alpha^2 - 3\rho + 2n\rho)g(X_2, X_3)\\ &+ (10\alpha^2 - 6n\alpha^2 + 3n\rho - 4\rho)\eta(X_2)\eta(X_3).\end{aligned} \tag{7.6}$$

Thus we have the following:

Theorem 7.1. *A conharmonically flat $(\mathcal{L}cs)_n$-manifold with $\mathcal{SVK}$-connection is an η-Einstein manifold with the Levi-Civita connection.*

Again using (7.6) in (4.9), we obtain

$$\bar{\mathcal{S}}(X_2, X_3) = 2(2-n)(\alpha^2 - \rho)g(X_2, X_3) + 2(3-2n)(\alpha^2 - \rho)\eta(X_2)\eta(X_3). \tag{7.7}$$

Thus, we state

Theorem 7.2. *A conharmonically flat $(\mathcal{L}cs)_n$-manifold with $\mathcal{SVK}$-connection is an η-Einstein manifold.*

8. Concircularily flat $(\mathcal{L}cs)_n$-manifolds with $\mathcal{SVK}$-connection

The concircular curvature tensor $\breve{C}$ with respect to the $\mathcal{SVK}$-connection $\bar{\bigtriangledown}$ is defined as

$$\breve{C}(X_1, X_2)X_3 = \bar{\mathcal{R}}(X_1, X_2)X_3 - \frac{\bar{r}}{n(n-1)}[g(X_2, X_3)X_1 - g(X_1, X_3)X_2] \tag{8.1}$$

for all X_1, X_2, X_3 in $\mathcal{M}$.
Now assume that $(\mathcal{L}cs)_n$-manifold is concircularily flat with $\mathcal{SVK}$-connection then

$$\breve{C}(X_1, X_2)X_3 = 0 \tag{8.2}$$

It follows

$$\bar{\mathcal{R}}(X_1, X_2)X_3 = \frac{\bar{r}}{n(n-1)}[g(X_2, X_3)X_1 - g(X_1, X_3)X_2] \tag{8.3}$$

Using (2.10),(2.12), (4.6) and (4.12), we get

$$\left[\frac{r-(n-1)(4\alpha^2+3n\alpha^2+2\rho)}{n(n-1)}\right][g(X_2, X_3)\eta(X_1) - g(X_1, X_3)\eta(X_2)] = 0. \tag{8.4}$$

This implies that either scalar curvature of the manifold is $r = (n-1)(4\alpha^2+3n\alpha^2+2\rho)$ or

$$g(X_2, X_3)\eta(X_1) - g(X_1, X_3)\eta(X_2) = 0. \tag{8.5}$$

Put $X_1 = \xi$ in (8.5) and using (2.10)

$$g(X_2, X_3) = -\eta(X_2)\eta(X_3) \tag{8.6}$$

Replace X_3 by QX_3, we have

$$S(X_2, X_3) = -(n-1)(\alpha^2 - \rho)\eta(X_2)\eta(X_3). \tag{8.7}$$

Hence, we state a theorem.

Theorem 8.1. *For a concircularly flat $(\mathcal{L}cs)_n$-manifold with $\mathcal{SVK}$-connection, either the scalar curvature is $(n-1)(4\alpha^2+3n\alpha^2+2\rho)$ or the manifold is special type of η-Einstein manifold.*

9. An example of $(\mathcal{L}cs)_n$-manifolds with $\mathcal{SVK}$-connection

Considering a 3-dimensional smooth manifold $\mathcal{M}=\left\{(x, y, z) \in R^3 : z \neq 0\right\}$, with (x, y, z), the standard coordinates. Let $\{e_1, e_2, e_3\}$ be linearly independent global frame on $\mathcal{M}$ given by

$$e_1 = z^2\frac{\partial}{\partial x}, \quad e_2 = z^2\frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z} \tag{9.1}$$

Let g be the Lorentzian metric defined by

$$g(e_1, e_1) = g(e_2, e_2) = 1, \quad g(e_3, e_3) = -1, \quad g(e_1, e_2) = g(e_2, e_3) = g(e_3, e_1) = 0$$

and η be 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z \in \chi(\mathcal{M})$.
Considering a (1,1) tensor field ϕ defined by

$$\phi(e_1) = -e_2, \quad \phi(e_2) = -e_1, \quad \phi(e_3) = 0.$$

Then using linearity of g and ϕ, we get

$$\begin{aligned} \eta(e_3) &= -1, \\ \phi^2(Z) &= Z + \eta(Z)e_3, \\ g(\phi Z, \phi W) &= g(Z, W) + \eta(Z)\eta(W). \end{aligned}$$

for any Z, $W \in \chi\ (\mathcal{M})$. Now, by computation directly, we get

$$[e_1, e_3] = -\frac{2}{z}e_1, \quad [e_1, e_3] = -\frac{2}{z}e_1, \quad [e_1, e_2] = 0.$$

Using Koszul's formula for the Lorentzian metric g, we have

$$\begin{aligned} \bigtriangledown_{e_1} e_3 = -\frac{2}{z}e_1, \quad \bigtriangledown_{e_2} e_3 = -\frac{2}{z}e_2, \quad \bigtriangledown_{e_3} e_3 = 0, \\ \bigtriangledown_{e_1} e_2 = 0, \quad \bigtriangledown_{e_2} e_2 = \frac{2}{z}e_3, \quad \bigtriangledown_{e_3} e_2 = 0, \\ \bigtriangledown_{e_1} e_1 = \frac{2}{z}e_3, \quad \bigtriangledown_{e_2} e_1 = 0 \quad \bigtriangledown_{e_3}\ e_1 = 0. \end{aligned} \tag{9.2}$$

From the above it can be easily seen that $e_3 = \xi$ is a unit timelike concircular vector field and hence $(\phi, \xi, \eta, g, \alpha)$ is an $(Lcs)_3$-structure on $\mathcal{M}$. Consequently $\mathcal{M}^3(\phi, \xi, \eta, g, \alpha)$ is an $(Lcs)_3$-manifold with $\alpha = -\frac{2}{z} \neq 0$ such that $(X\alpha) = \rho\eta(X)$ where $\rho = \frac{2}{z^2}$.
Using the above relations, we calculate components of curvature tensor R as follows:

$$\begin{aligned} R(e_1, e_3)e_3 = -\frac{6}{z^2}e_1, \quad R(e_2, e_3)e_3 = -\frac{6}{z^2}e_2, \\ R(e_i, e_j)e_3 = 0, \quad R(e_i, e_j)e_j = -\frac{4}{z^2}e_i, \ i, j = 1, 2 \\ R(e_1, e_3)e_2 = 0, \quad R(e_2, e_3)e_1 = 0, \quad R(e_3, e_i)e_i = -\frac{6}{z^2}e_3. \end{aligned} \tag{9.3}$$

Now the $\mathcal{SVK}$-connection on $\mathcal{M}$ is given as

$$\begin{aligned} \bar{\nabla}_{e_1} e_3 = -(\frac{2}{z}+\alpha)e_1, \quad \bar{\nabla}_{e_2} e_3 = -(\frac{2}{z}+\alpha)e_2, \quad \bar{\nabla}_{e_3} e_3 = \alpha(e_3-\xi), \\ \bar{\nabla}_{e_1} e_2 = 0, \quad \bar{\nabla}_{e_2} e_2 = \frac{2}{z}e_3+\alpha\xi, \quad \bar{\nabla}_{e_3} e_2 = 0, \\ \bar{\nabla}_{e_1} e_1 = \frac{2}{z}e_3+\alpha\xi, \quad \bar{\nabla}_{e_2} e_1 = 0 \quad \bar{\nabla} n_{e_3} e_1 = 0. \end{aligned} \tag{9.4}$$

From (9.4) we have that $\bar{\nabla}_{e_i} e_j = 0$ $(1 \leq i, j \leq 3)$ for $\xi = e_3$ and $\alpha = -\frac{2}{z}$. Hence $\mathcal{M}$ is a 3-dimensional $(\mathcal{L}cs)_n$-manifold with$\mathcal{SVK}$-connection.

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